MULTI-VALUED MONOTONE NONLINEAR MAPPINGS AND DUALITY MAPPINGS IN BANACH SPACES

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Introduction. Let X be a reflexive real Banach space, X^* its conjugate space, (w, u) the pairing between w in X^* and u in X. We consider multi-valued mappings T of X into X^* (i.e , mappings in the ordinary sense of X into 2^{X^*}) which are monotone, i.e., if $v \in T(u)$, $v_1 \in T(u_1)$ for u and u_1 in X, then

$$(v - v_1,\ u - u_1) \geqq 0.$$

It is our object in the present paper to generalize to the multi-valued case the results obtained in a number of recent papers by the author and G. J. Minty for single-valued mappings T (cf. [2]–[14]). The first results for multi-valued mappings for X a Hilbert space have been obtained in an unpublished paper of Minty [15]. The methods of [15] are not directly extendable to more general spaces, but our discussion of the finite-dimensional case (Lemma 2.1) has been very much influenced by the manuscript of [15] which Minty has recently transmitted to the author. (The basic result of [15] is stated at the end of §2 below.)

Our results for general multi-valued monotone mappings have an interesting specific application given in §3 below to the generalization of a theorem of Beurling and Livingston [1] on duality mappings in Banach spaces. In a previous paper [12], we showed that for strictly convex reflexive spaces, this theorem could be obtained from results on single-valued monotone mappings. In §3 below we give a generalization of this theorem to general reflexive Banach spaces which runs as follows: *Let X be a reflexive Banach space, $\phi(r)$ a non-negative non-decreasing function on R^1 with $\phi(0) = 0$. The duality map T of X with respect to ϕ is defined by*

$$T(u) = \begin{cases} v \mid v \in X^*, & \|v\| = \phi(\|u\|), \\ (v,u) = \|v\| \cdot \|u\|. \end{cases}$$

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Let Y be a closed subspace of X, $Y^{\perp}$ its annihilator in X^*, v_0 and w_0 arbitrary elements of X and X^*, respectively. Then

$$T(Y+v_0)\cap(Y^{\perp}+w_0)$$

is nonempty.

§1 is devoted to the study of maximal monotonic mappings and of a very weak continuity property for multi-valued mappings which we have called *vague continuity* and which plays a key role in our discussion. §2 contains the proof of the basic results on general multi-valued monotonic mappings. §3 contains the discussion of duality mappings.

1. Let X be a reflexive Banach space over the reals, X^* its conjugate space. We denote the pairing between w in X^* and u in X by (w,u). We denote by $X \times X^*$ the product space of X and X^* whose elements will be written as $[u,w]$ and with norm

$$\| [u,w] \| = \{\| u \|_X^2 + \| w \|_{X^*}^2\}^{1/2}.$$

We consider multi-valued mappings T of X into X^*, where T assigns to each u in X, a subset $T(u)$ (possibly empty) of X^*.

To make our discussion of multi-valued mappings more intuitive by tying the formalism of our arguments closer to the single-valued case, we introduce the following notational convention:

CONVENTION. *If V is a subset of X^*, u an element of X, then (V,u) will denote the set $\{(v,u) \mid v\in V\}$. Similarly if V and W are subsets of X^*, then $(V-W,u)$ will denote the set $\{(v-w,u) \mid v\in V, w\in W\}$. If c is a real number, and R_0 is a set of real numbers, $R_0 \geqq c$ (or $R_0 \leqq c$) will denote the sets of inequalities $r \geqq c$ for $r\in R_0$ (or $r\leqq c$ for $r\in R_0$). If a set V appears several times in a single equation or inequality, the equation or inequality is assumed to hold for each v in V, with the same v chosen at all points of occurrence of V in the given equation or inequality.*

DEFINITION 1.1. *Let T be a (possibly) multi-valued map from X to X^*. Then T is said to be monotone if*

$$(T(u)-T(u_1),\ u-u_1)\geqq 0$$

for all u and u_1 in X.

DEFINITION 1.2. *The graph G(T) is the subset of $X\times X^*$ given by*

$$G(T) = \{[u,w] \mid w\in T(u),\ u\in X\}.$$

We say that $T\subseteq T_1$ if $G(T)\subseteq G(T_1)$.

DEFINITION 1.3. *T is said to be maximal monotone if T is monotone and if for every monotone T_1 such that $T\subseteq T_1$, we have $T=T_1$.*

If S is a subset of X or X^*, $K(S)$ will denote its convex closure, i.e., the smallest closed convex set containing S. S is said to surround 0 if every ray $\{tw \mid t > 0\}$ for $w \neq 0$ intersects S.

LEMMA 1.1. *Let T be a maximal monotone multi-valued map from X to X^*. Then:*

(a) *For every u in $X, T(u)$ is a closed convex subset of X^*.*

(b) *If $\{u_k\}$ and $\{v_k\}$ are sequences in X and X^*, respectively, such that $u_k \to u_0$ strongly in $X, v_k \in T(u_k)$, and $v_k \to v_0$ weakly in X^*, then $v_0 \in T(u_0)$.*

Proof of Lemma 1.1. Proof of (a). For u, u_1 in X and $v, v_0 \in T(u)$, $v_1 \in T(u_1)$, we have

$$(v - v_1, u - u_1) \geqq 0,$$

$$(v_0 - v_1, u - u_1) \geqq 0.$$

If $0 \leqq t \leqq 1$, $v_t = tv + (1-t)v_0$, we have

$$(v_t - v_1, u - u_1) = t(v - v_1, u - u_1) + (1 - t)(v_0 - v_1, u - u_1) \geqq 0.$$

If we add v_t to $T(u)$ therefore to obtain a larger mapping T_1, it follows that T_1 is monotone. Since T is maximal monotone, it follows that $v_t \in T(u)$, i.e., $T(u)$ is convex. Similarly $T(u)$ is closed.

Proof of (b). Let u be any element of X, v any element of $T(u)$. For every k, we have

$$(v_k - v, u_k - u) \geqq 0.$$

Since $u_k - u$ converges strongly to $u_0 - u$ while $v_k - v$ converges weakly to $v_0 - v$, we have

$$(v_k - v, u_k - u) \xrightarrow[k \to \infty]{} (v_0 - v, u_0 - u).$$

Hence

$$(v_0 - v, u_0 - u) \geqq 0$$

for every u in X, $v \in T(u)$. By the maximal monotonicity of T, it follows that $v_0 \in T(u)$. Q.E.D.

DEFINITION 1.4. *If T is a multi-valued transformation from X to X^*, its domain $D(T)$ is defined to be the set of u in X for which $T(u) \neq \emptyset$.*

DEFINITION 1.5. *If T is a multi-valued mapping from X to X^*, T is said to be vaguely continuous if $D(T)$ is a dense linear subset of X and the following condition is satisfied.*

For each pair u_0 and u_1 of $D(T)$, there exists a sequence $\{t_n\}$ of positive real numbers with $t_n \to 0$ as $n \to +\infty$ and an element v_1 of $K(T(u_0))$ such that if $u_n = t_n u_1 + (1 - t_n)u_0$, there exist elements $v_n \in K(T(u_n))$ such that $v_n \to v_1$ weakly in X^*.

If T is a single-valued mapping, vague continuity of T is a weakening of the condition of hemi-continuity of T as introduced by the author in [5] (i.e., T continuous from each segment in $D(T)$ to the weak topology of X^*),

THEOREM 1.1. *Let T be a maximal monotone mapping of X into X^* such that $D(T)$ is a dense linear subset of X and for each closed line segment S_0 in $D(T)$, there is a bounded set S_1 in X^* such that $T(u) \cap S_1 \neq \emptyset$ for $u \in S_0$.*

Then T is vaguely continuous and $T(u)$ is a closed convex set for every u in $D(T)$.

Proof of Theorem 1.1. We know from the maximal monotonicity of T and part (a) of Lemma 1.1 that $T(u)$ is a closed convex set in X^* for every u in $D(T)$. It follows from the hypotheses of our theorem that $D(T)$ is a dense linear subset of X. We need only to show that the condition of Definition 1.5 is satisfied.

Let S_0 be the closed line segment $\{u_t = tu_1 + (1-t)u_0 \mid 0 \leqq t \leqq 1\}$ in $D(T)$. By hypothesis, there exists a constant M depending on S_0 such that for each u_t in S, we may find v_t in $T(u_t)$ with $\|v_t\| \leqq M$. By the weak compactness of the closed ball in the reflexive Banach space X^*, we may choose a sequence $\{t_n\}$ with $t_n > 0$, $t_n \to 0$ as $n \to +\infty$ and $v_{t_n} \to v_1$ weakly in X^* as $n \to +\infty$. However, $u_{t_n} \to u_0$ strongly in X. Since T is maximal monotone, it follows from Lemma 1.1(b) that $v_1 \in T(u_0)$. Q.E.D.

We have a converse for Theorem 1.1, namely:

THEOREM 1.2. *Let T be a multi-valued mapping of X into X^* for which all of the following conditions are satisfied.*

(a) *T is monotone.*

(b) *$D(T) = X$ and $T(u)$ is a closed convex set for each u in X.*

(c) *T is vaguely continuous.*

Then T is maximal monotone.

Proof of Theorem 1.2. Suppose $T \subseteq T_1$, where T_1 is monotone and $v_0 \in T_1(u_0)$. We must show that $v_0 \in T(u_0)$. By the monotonicity of T_1, we know that for every u in X and $v \in T(u)$, we have

$$(v - v_0,\ u - u_0) \geqq 0.$$

Suppose v_0 does not lie in $T(u_0)$. Since $T(u_0)$ is closed and convex there exists w in X such that

$$(v_0, w) > (T(u_0), w).$$

For real $t > 0$, set $u_t = u_0 + tw$. For any v in $T(u_t)$, we have

$$t(v - v_0, w) \geqq 0,$$

i.e.,

$$(v - v_0, w) \geqq 0, \qquad v \in T(u_t),$$

or

$$(T(u_t) - v_0, w) \geqq 0.$$

Hence

$$(T(u_t) - T(u_0), w) \geqq (v_0 - T(u_0), w)$$

for all $t > 0$. Hence, choosing $\{v_k\}$ for the segment $\{u_t = u_0 + tw \mid 0 \leqq t \leqq 1\}$ we have $v_k \in T(u_k)$, where $u_k = u_0 + t_k w$ $(t_k \to 0)$ with $v_k \to v_1$ weakly in X^* for some v_1 in $T(u_0)$. Hence

$$(v_k - v_1, w) \geqq (v_0 - v_1, w),$$

which implies that

$$0 \geqq (v_0 - v_1, w) \geqq (v_0 - T(u_0), w) > 0,$$

yielding a contradiction. Q.E.D.

LEMMA 1.2. *If T is a maximal monotone multi-valued mapping from X to X^* and if for sequences $\{u_k\}$ and $\{v_k\}$ from X and X^*, respectively, we have*

$$v_k \in T(u_k)$$

and

$$u \to g_0 \quad \textit{weakly in } X,$$

$$v_k \to v_0 \quad \textit{strongly in } X^*,$$

then $v_0 \in T(u_0)$.

Proof of Lemma 1.2. For u in $X, v \in T(u)$, we have for every k

$$(v_k - v, u_k - u) \geqq 0.$$

Since $u_k - u$ converges weakly to $u_0 - u$ and $v_k - v$ converges strongly to $v_0 - v$, we have

$$(v_k - v, u_k - u) \to (v_0 - v, u_0 - u).$$

Hence,

$$(v_0 - v, u_0 - u) \geqq 0,$$

i.e.,

$$(v_0 - T(u), u_0 - u) \geqq 0$$

for all u in X. By the maximal monotonicity of T, it follows that $v_0 \in T(u_0)$. Q.E.D.

2. We begin the study of the ranges of monotone multi-valued mappings with the finite-dimensional case.

LEMMA 2.1. *Let F be a finite-dimensional Banach space, F^* its conjugate space, T a multi-valued mapping of F into F^*. Suppose that T is maximal* *monotone and that there exists a bounded subset S of F surrounding* 0 *such that for u in S,*

$$(T(u), u) \geqq 0.$$

Then there exists u_0 in $K(S)$ such that $0 \in T(u_0)$.

Proof of Lemma 2.1. Since the hypotheses and conclusions are invariant under a change to an equivalent norm and since F is of finite dimension, we may assume without loss of generality that F is a Hilbert space and $F^* = F$.

We adopt a device used by Minty [15] under different hypotheses in infinite-dimensional Hilbert spaces. For each positive integer n, let T_n be the mapping from X to X^* whose graph is given by

$$G(T_n) = \left\{ \left[u + \frac{1}{n} v, v + \frac{1}{n} u \right] \mid [u, v] \in G(T) \right\}.$$

We consider the properties of the mappings T_n. We begin by establishing the inequality:

$$(w - w_1, x - x_1) \geqq \frac{1}{4n} \{ \| w - w_1 \|^2 + \| x - x_1 \|^2 \} \tag{2.1}$$

for all $[x, w]$ and $[x_1, w_1]$ in $G(T_n)$. By the definition of $G(T_n)$, there exist $[u, v]$ and $[u_1, v_1]$ in $G(T)$ such that

$$x = u + \frac{1}{n} v, \qquad w = v + \frac{1}{n} u,$$

$$x_1 = u_1 + \frac{1}{n} v_1, \qquad w_1 = v_1 + \frac{1}{n} u_1.$$

Hence,

$$\begin{aligned}(w - w_1, x - x_1) &= \left((u - u_1) + \frac{1}{n}(v - v_1), (v - v_1) + \frac{1}{n}(u - u_1) \right) \\ &\geqq \frac{1}{n} \{ \| u - u_1 \|^2 + \| v - v_1 \|^2 \}\end{aligned}$$

On the other hand,

$$\| x - x_1 \| \leqq \| u - u_1 \| + \| v - v_1 \|,$$

$$\| w - w_1 \| \leqq \| u - u_1 \| + \| v - v_1 \|$$

so that

$$\| x - x_1 \|^2 + \| w - w_1 \|^2 \leqq 4 \{ \| u - u_1 \|^2 + \| v - v_1 \|^2 \}$$

and

$$(w - w_1, x - x_1) \geqq \frac{1}{4n}\{\| x - x_1 \|^2 + \| w - w_1 \|^2\}.$$

As a corollary of the inequality (2.1), we see that if $x = x_1$, then $w = w_1$ and conversely so that T_n is a one-to-one mapping with

$$\frac{1}{4n} \| x - x_1 \| \leqq \| T_n x - T_n x_1 \| \leqq 4n \| x - x_1 \|.$$

If T is maximal monotone, the transformation $T^{\#}$ with graph

$$G(T^{\#}) = \left\{ \left[u, \frac{v}{n} \right] \mid [u,v] \in G(T) \right\}$$

is also maximal monotone. Applying Lemma 2 of Minty [13], we see that the set $\{u + v/n \mid [u,v] \in G(T)\}$ is the whole of F. Hence each T_n is defined on all of X and satisfies the inequality

$$(T_n x - T_n x_1, x - x_1) \geqq \frac{1}{4n} \| x - x_1 \|^2.$$

Hence by [13], each T_n maps F one-to-one *onto* F.

For each n, let x_n be the unique solution of $T_n x_n = 0$. Choose $[u_n, v_n] \in G(T)$ such that

$$u_n + \frac{1}{n} v_n = x_n,$$

$$v_n + \frac{1}{n} u_n = 0.$$

We assert that $u_n \in K(S)$. Indeed for u not in $K(S)$, we have $u = \rho u_0$, where $\rho > 1$, $u_0 \in S$ (since S surrounds the origin). Since

$$(T(u) - T(u_0),\ u - u_0) \geqq 0$$

we have for $v \in T(u_0)$,

$$\frac{(\rho - 1)}{\rho}(T(u), u) \geqq (\rho - 1)(Tu_0, u_0) \geqq 0,$$

i.e., for $v \in T(u)$, $(v, u) \geqq 0$. For such u and v

$$\left(v + \frac{1}{n}\, u, v \right) \geqq \| v \|^2,$$

$$\left(v + \frac{1}{n}\, u, u \right) \geqq \frac{1}{n} \| u \|^2$$

so that if $v + (1/n)u = 0$, we have $u = 0, v = 0$, i.e., $u \in K(S)$, which is a contradiction. Hence all the elements u_n lie in $K(S)$.

Since $K(S)$ is bounded, there exists a constant M such that $\| u_n \| \leqq M$ for all n. Hence

$$\| v_n \| = \left\| \frac{1}{n} u_n \right\| \leqq \frac{M}{n}$$

so that $v_n \to 0$ as $n \to \infty$. We may choose a subsequence $\{u_{n_j}\}$ so that $u_{n_j} \to u_0$ in F as $j \to +\infty$. By Lemma 2.1, however, it follows that $0 \in T(u_0)$. Q.E.D.

LEMMA 2.2. *Let T be a multi-valued mapping of X into X^* such that*

(a) *T is monotone.*

(b) *T is vaguely continuous.*

(c) *$T(u)$ is a bounded closed convex set for each u.*

Let Y be a closed subspace of X such that $Y \subset D(T)$. Let j be the injection mapping of Y into X, j^* the projection map of X^* onto Y^*. Let T_1 be the multi-valued mapping of Y into Y^* given by $T_1(u) = j^*T(ju)$ for u in Y.

Then T_1 is monotone, $D(T_1) = Y$, and T_1 satisfies conditions (a), (b), *and* (c). *In particular, T_1 is maximal monotone.*

Proof of Lemma 2.2. For each u in Y, $T(u) \neq \emptyset$ implies that $T_1(u) \neq \emptyset$. Hence $D(T_1) = Y$.

For u, u_1 in Y

$$(T_1(u) - T_1(u_1), u - u_1) = (T(u) - T(u), u - u_1) \geqq 0$$

so that T_1 is monotone.

Since j^* is weakly continuous, if $v_k \in T(u_k)$ and $v_k \to v_1$ weakly in X^* for $v_1 \in T(u_0)$, then $j^*v_k \in T_1(u_k)$, $j^*v_1 \in T_1(u_0)$, and $j^*v_k \to j^*v_1$ weakly in Y^*. Hence T_1 is vaguely continuous.

Since j^* is linear and $T(u)$ is convex for each $u, j^*T(u) = T_1(u)$ is convex for each u in Y. Since $T(u)$ is a bounded closed convex set in the reflexive space X^*, it is weakly compact. Since j^* is weakly continuous, $j^*T(u) = T_1(u)$ is weakly compact and hence closed. Thus we have completed the verification of properties (a), (b), and (c) for the mapping T_1.

Finally the maximal monotonicity of T_1 follows from (a), (b), and (c) and Theorem 1.2. Q.E.D.

THEOREM 2.1. *Let T be a multi-valued mapping of X into X^* such that $T(u)$ is bounded for each u, $D(T)$ is a linear subset of X, and for each closed line segment S_0 in $D(T)$, there exists a bounded set S_1 in X^* (possibly depending on S_0) such that $T(u) \cap S_1 \neq \emptyset$ for $u \in S_0$. Suppose further that*

(i) *T is maximal monotone.*

(ii) *There exists a bounded subset S of X surrounding* 0 *such that*

$$(T(u), u) \geqq 0$$

for $u \in S$.

Then there exists u_0 *in* $K(S)$ *such that* $0 \in T(u_0)$.

Proof of Theorem 2.1. Since T is maximal monotone and a bounded set S_1 exists for each closed line segment S_0 such that $T(u) \cap S_1 \neq \varnothing$ for $u \in S_0$, it follows from Theorem 1.1 that T is vaguely continuous, and $T(u)$ is a bounded closed convex subset of X^* for each u in $D(T)$.

Let F be a finite-dimensional subspace of $D(T)$. Let j_F be the injection mapping of F into X, j_F^* the dual map projecting X^* onto F^*. We form the mapping $T_F: F \to F^*$ by setting $T_F u = j_F^*(T_F(j_F u))$ $(u \in F)$. Then by Lemma 2.2, T_F is vaguely continuous, $T_F(u)$ is a closed convex subset of F^* for every u in F, $D(T_F) = F$, and T_F is a monotone multi-valued mapping of F into F^*. Hence by Theorem 1.2, T_F is a maximal monotone mapping of F into F^*.

Let $S_F = S \cap F$. Then $S_F \subset K(S_F) \subset K(S)$, and S_F surrounds the origin in F. For u in S_F,

$$(T_F(u), u) = (j_F^* T(u), u) = (T(u), u) \geqq 0.$$

Hence T_F satisfies the hypotheses of Lemma 2.1 and there exists u_F in $K(S_F) \subset K(S) \cap F$ such that $0 \in T_F(u_F)$.

For any u in F, we have, however,

$$(T_F(u_F) - T_F(u), u_F - u) \geqq 0,$$

i.e.,

$$(T(u), u - u_F) \geqq 0.$$

Hence, the set

$$V_F = \{v \mid v \in K(S),\ (T(u), u - v) \geqq 0\} \quad \text{for all } u \in F$$

is a nonempty weakly closed convex subset of the weakly compact set $K(S)$ in X. Since the family of such sets is closed under finite intersections, it follows that the set

$$\bigcap_F V_F \neq \varnothing.$$

If u_0 lies in $\bigcap_F V_F$, however, u_0 lies in $K(S)$, and

$$(T(u), u - u_0) \geqq 0$$

for all $u \in D(T)$. Hence by the maximal monotonicity of $T, 0 \in T(u_0)$. Q.E.D.

THEOREM 2.2. *Let* T *be a multi-valued mapping of* X *into* X^* *such that* $D(T) = X$, T *is monotone and vaguely continuous, and* $T(u)$ *is a bounded closed convex set for each* u. *Suppose that there exists a bounded set* S *surrounding* 0 *in* X *such that* $(T(u), u) \geqq 0$ *for* u *in* S.

Then there exists u_0 *in* $K(S)$ *such that* $0 \in T(u_0)$.

Proof of Theorem 2.2. This is the same as that of Theorem 2.1 except that the vague continuity of T is given to us by hypothesis and does not need to be deduced from maximal monotonicity and the existence of sets S_1 as in Theorem 2.1.

THEOREM 2.3. *Let T be a monotone multi-valued mapping of X into X^* Y a closed subspace of X, $Y^\perp$ its annihilator in X^*. Suppose that $Y \subset D(T)$ and that there exists a subset S surrounding* 0 *in Y such that $(T(u),u) \geqq 0$ for u in S. Suppose also that one of the two following conditions holds*:

(A) *T is maximal monotone. $T(u)$ is a bounded set for each u, and for each closed segnent S_0 in X, there exists a bounded set S_1 in X^* such that $T(u) \cap S_1 \neq \varnothing$.*

(B) *T is vaguely continuous and $T(u)$ is a bounded closed convex subset of X^* for each u.*

Then there exists u_0 in $K(S) \subset Y$ such that $T(u_0) \cap Y^\perp \neq \varnothing$.

Proof of Theorem 2.3. If j is the injection mapping of Y into X, j^* the projection mapping of X^* on Y^*, we set $T_1(u) = j^*(T(u))$. Then $T(u_0) \cap Y^\perp \neq \varnothing$ if and only if $0 \in T_1(u)$. If (A) holds, T_1 satisfies the hypotheses of Theorem 2.1, while if (B) holds, T_1 satisfies the hypotheses of Theorem 2.2. Hence our conclusion follows. Q.E.D.

THEOREM 2.4. *Let T be a monotone multi-valued mapping of X into X^*, Y a closed subspace of X with $Y \subset D(T)$, $Y^\perp$ the annihilator of Y in X^*. Suppose that T satisfies either of the conditions* (A) *and* (B) *of Theorem* 2.3 *and that there exists a continuous real-valued function on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that*

$$(T(u),u) \geqq c(\|u\|)\{\|u\| + \|T(u)\|\}$$

for $u \in Y$.

Then for each v_0 in X, w_0 in X^*,

$$T(Y + v_0) \cap (w_0 + Y^\perp) \neq \varnothing.$$

Proof of Theorem 2.4. We form the mapping $T^{\#}$ of X into X^* by setting

$$T^{\#}(u) = T(u_0 + v_0) - w_0.$$

Then $T^{\#}$ satisfies the hypotheses of Theorem 2.3 with respect to Y since for $\|u\|$ sufficiently large

$$\begin{aligned}(T(u+v_0) - w_0, u) &= (T(u+v_0), u+v_0) - (w_0,u) - (T(u+v_0), v_0) \\ &\geqq c(\|u+v_0\|)\{\|u+v_0\| + \|T(u+v_0)\|\} - \|w_0\| \cdot \|u\| \\ &\quad - \|v_0\| \cdot \|T(u+v_0)\| \geqq 0. \qquad \text{Q.E.D.}\end{aligned}$$

It is interesting to compare Theorem 2.3 with the result obtained by Minty in [15]. In our notation, this is the following:

THEOREM (MINTY). *Let H be a Hilbert space, T a multi-valued mapping of H into H, Y a closed subspace of H. Suppose that T is maximal monotone and satisfies all of the following conditions:*

(i) $(T(u),u) \geqq -c$ *for some* $c > 0$ *and all* u *in* H.

(ii) *There exists a bounded set C surrounding 0 in H such that for every u in C, there exists $v \in T(u)$ such that*

$$(v,u) \geqq 0.$$

(iii) *There exists a bounded set D in H surrounding 0 such that for each $v \in D$, there exists u in H such that $v \in T(u)$ and*

$$(v,u) \geqq 0.$$

Then $T(X) \cap Y^{\perp} \neq \emptyset$.

To clarify the relation of this result to Theorem 2.3, we note that by the monotonicity of T, the condition (ii) of Minty's theorem is equivalent to the stronger condition:

(ii)′ $C \subset D(T)$ *and* $(Tu),u) \geqq 0$ *for* $u \in C$.

Indeed if $k > 1$ is fixed and $u \in C$, we have from condition (ii):

$$0 \leqq (T(ku) - v, ku - u) = (k-1)\left\{\frac{1}{k}(T(ku), ku) - (v,u)\right\}.$$

Hence if $u_1 = ku \in kC, (T(u_1), u_1) \geqq 0$.

Theorem 2.4 is thus a generalization of Minty's theorem to reflexive Banach spaces with hypotheses (i) and (iii) dropped and with the additional hypotheses that $T(u)$ is bounded for each u and that for each line segment S_0, there exists a bounded set S_1 intersecting $T(u)$ for all u in S_0,

3. Let X be a reflexive Banach space as before, X^* its conjugate space, ϕ a continuous nondecreasing non-negative function of r in R^1 with $\phi(0) = 0$, $\phi(r) \to +\infty$ as $r \to +\infty$.

DEFINITION. *If $u \neq 0$ is an element of X, v in X^* is said to be a dual element to u with respect to the gauge function ϕ if*

$$(v,u) = \|v\| \cdot \|u\|,$$
$$\|v\| = \phi(\|u\|).$$

DEFINITION. *The duality map T of X into X^* (with respect to the gauge function ϕ) is given by $T(0) = 0$ and for $u \neq 0$,*

$$T(u) = \{v \mid v \text{ is dual to } u\}.$$

LEMMA 3.1. *If X is a reflexive Banach space, ϕ a continuous non-negative nondecreasing function on R^1 with $\phi(0) = 0$, then the duality map T of X into X^* with respect to ϕ is a multi-valued maximal monotone mapping of X into X^* with $D(T) = X$ and*

(a) *T is vaguely continuous.*

(b) *$T(u)$ is a bounded closed convex subset of X^* for each u in X.*

(c) *For all u in X*

$$(T(u), u) \geqq c(\|u\|)\{\|u\| + \|Tu\|\},$$

where

$$c(r) = \min\left\{\frac{1}{2}r, \frac{1}{2}\phi(r)\right\}.$$

Proof of Lemma 3.1. The maximal monotonicity of T will follow if we prove that T is monotone, $D(T) = X$, and (a), (b), and (c) above are valid. $D(T) = X$ by the Hahn-Banach theorem. If $u, u_1 \in X$ and $v \in T(u)$, $v_1 \in T(u_1)$, then

$$\begin{aligned}(v - v_1, u - u_1) &= \|v\| \cdot \|u\| + \|v_1\| \cdot \|u_1\| - (v, u_1) - (v_1, u)\\ &\geqq \|v\| \cdot \|u\| + \|v_1\| \cdot \|u_1\| - \|v\| \cdot \|u_1\| - \|v_1\| \cdot \|u\|\\ &= (\|v\| - \|v_1\|)(\|u\| - \|u_1\|)\\ &= (\phi(\|u\|) - \phi(\|u_1\|))(\|u\| - \|u_1\|) \geqq 0,\end{aligned}$$

since ϕ is nondecreasing. Hence T is monotone.

Proof of (a). Let $\{u_k\}$ be a sequence converging strongly to $u_0, v_k \in T(u_k)$. Then $\|v_k\| = \phi(\|u_k\|) \leqq M$, so that by extracting a subsequence, we can assume that $v_k \to v_1$ weakly in X^*. Since $u_k \to u_0$ strongly, we have

$$\|v_k\| \cdot \|u_k\| = (v_k, u_k) \to (v_1, u_0)$$

while

$$\begin{aligned}\|v_1\| &\leqq \liminf \|v_k\|,\\ \|u_0\| &= \lim \|u_k\|.\end{aligned}$$

Hence

$$\|v_1\| \cdot \|u_0\| \leqq (v_1, u_0) \leqq \|v_1\| \cdot \|u_0\|.$$

Thus

$$(v_1, u_0) = \|v_1\| \cdot \|u_0\|.$$

Moreover

$$(v_1, u_0) = \lim (v_k, u_k) = \lim \phi(\|u_k\|)\|u_k\| = \phi(\|u_0\|)\|u_0\|$$

so that

$$\|v_1\| = \phi(\|u_0\|).$$

Thus $v_1 \in T(u_0)$.

Proof of (b). Obviously $T(u)$ is bounded and closed. Suppose $v, v_1 \in T(u)$ Then for $0 \leqq t \leqq 1$,

$$\begin{aligned}(tv + (1-t)v_1, u) &= t(v,u) + (1-t)(v_1,u)\\ &= t\phi(\| u \|) \| u \| + (1-t)\phi(\| u \|) \| u \|\\ &= \phi(\| u \|) \| u \|.\end{aligned}$$

However, if $v_t = tv + (1-t)v_1$, we have

$$\| v_t \| \leqq t \| v \| + (1-t) \| v_1 \| = \phi(\| u \|).$$

Hence

$$(v_t, u) = \phi(\| u \|) \| u \| \geqq \| v_t \| \ \| u \|$$

and since

$$(v_t, u) \leqq \| v_t \| \cdot \| u \|,$$

we have $\| v_t \| = \phi(\| u \|)$ and $v_t \in T(u)$. Hence $T(u)$ is convex. Q.E.D.

Proof of (c). For $u \in X$

$$\begin{aligned}(Tu, u) &= \phi(\| u \|) \| u \| = \tfrac{1}{2} \| T(u) \| \cdot \| u \| + \tfrac{1}{2}\phi(\| u \|) \ \| u \|\\ &\geqq c(\| u \|)\{\| u \| + \| T(u) \|\}. \qquad \text{Q.E.D.}\end{aligned}$$

THEOREM 3.1. *Let X be a reflexive Banach space, Y a closed subspace of X, X^* the conjugate space of X, $Y^\perp$ the annihilator of Y in X^*. Let T be a duality map of X into X^*. If $v_0 \in X, w_0 \in X^*$, then the set*

$$T(Y + v_0) \cap (Y^\perp + w_0) \neq \varnothing.$$

Proof of Theorem 3.1. By Lemma 3.1, T satisfies the hypotheses of Theorem 2.4 and our conclusion follows.
Q.E.D.

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